Int. J. Heat Mass Transfer. Vol. 7, pp. 327–332. Pergamon Press 1964. Printed in Great Britain

TEMPERATURE DISTRIBUTION IN COOLED TURBINE DISKS

A. BROWN† and E. MARKLAND‡

(*Received* 19 *August* 1963 *and in revised form* 17 *October* 1963)

Abstract—The axisymmetric distribution of temperature is calculated for disks in which the thickness varies exponentially with radius, neglecting temperature variation through the thickness. An analytical solution is found for the case of disks of uniform thermal conductivity, constant heat-transfer coefficient and negligible coolant temperature rise. When the conductivity varies with temperature, or the heat-transfer coefficient varies with radius, or the temperature rise of the coolant is appreciable, a numerical solution is adopted. A few computed results are presented, showing that relatively small flows of coolant produce effective cooling, and that variation of thermal conductivity with temperature is significant for some materials in current use.

NOMENCLATURE

$b = b_s r^n$,	thickness of disk at radius r;
f,	meridional length measured outwards along surface of disk;
h,	heat-transfer coefficient between disk surface and coolant;
$k = k_s(1 + \mu T)$,	thermal conductivity of disk material at temperature T;
r,	arbitrary radius;
c_p,	specific heat of coolant;
M,	mass flow rate of coolant over each side of disk;
T,	temperature of disk at radius r;
T_c,	temperature of coolant at radius r;
$s = (r - r_0)/(r_1 - r_0)$,	dimensionless radius;
$\phi = (T - T_{c0})/(T_1 - T_{c0})$,	dimensionless disk temperature;
$\theta = (T_c - T_{c0})/(T_1 - T_{c0})$,	dimensionless coolant temperature;
$A = r_0/(r_1 - r_0)$,	a parameter specifying the radial proportions of the disk;
$B = \frac{1}{2} b_s n (r_1 - r_0)^{n-1}$,	a parameter specifying the disk thickness;
$C = 2h(r_1 - r_0)^{2-n}/k_s b_s(1 + \mu T_{c0})$,	a parameter determined by heat-transfer coefficient and conductivity;
$D = \mu(T_1 - T_{c0})/(1 + \mu T_{c0})$,	a parameter determined by temperature coefficient of conductivity;
$E = 2\pi h(r_1 - r_0)^2/Mc_p$,	a coolant mass flow parameter;
$I\nu$, $K\nu$,	νth order modified Bessel functions of first and second kinds;

$$z = \frac{2}{2-n} C^{\frac{1}{2}} (A + s)^{1-n/2};$$

subscripts 0 and 1 refer to inner and outer radii of the disk.

† Department of Mechanical Engineering, The Queen's University of Belfast.
‡ Department of Mechanical Engineering, University of Nottingham.

INTRODUCTION

A GAS turbine disk, in which the thickness usually varies with radius, receives heat from the blades at its rim and may be cooled by forced convection of air, admitted at the hub, flowing outwards over the surfaces of the disk. The heat removed from the disk and the temperature distribution in it are important in determining the blade temperature and thermal stresses in the disk.

There appears to be surprisingly little published on temperature distributions in such disks. Johnson [1] suggests that the difference between temperature at any radius of a thin disk and that at the hub is proportional to the 4th power of radius, but the grounds for this statement are not known. Bert [2] has given brief details of a non-symmetric solution for a disk of constant thermal conductivities in the radial and tangential directions, over which the heat-transfer coefficient is an exponential function of radius. In the present work we deal with axisymmetric temperature distributions; an analytical solution is found for the case of thin disks of uniform thermal conductivity, constant heat-transfer coefficient, and negligible coolant temperature rise. In practical cases where these conditions are not met, numerical solutions may be found. Runge-Kutta's method proves useful in this instance because of its accuracy, adaptability and availability of a suitable subroutine in many digital computers.

ANALYTICAL SOLUTION OF THE HEAT FLOW EQUATIONS

Consider the disk of Fig. 1 in which the thickness b at any radius r between inner and outer radii r_0 and r_1 satisfies the relationship

$$b = b_s r^n \tag{1}$$

where b_s and n are constants. Heat is supplied uniformly to the disk at its periphery, where its temperature is T_1, and is removed by forced convection of a coolant, of specific heat c_p. This is introduced at a known temperature T_{c0} at the inner radius and flows outwards over both surfaces of the disk. The mass flow rate is M over each side of the disk. It is assumed that the disk is sufficiently thin for temperature variation in the direction of the axis to be negligible and

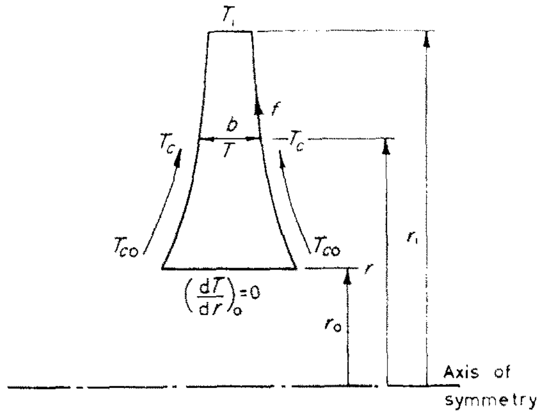

FIG. 1. Diagram showing notation.

that there is no radial heat flux at the inner radius, i.e.

$$\left(\frac{dT}{dr}\right)_0 = 0. \tag{2}$$

The heat-transfer coefficient h from the disk to the coolant is assumed to be constant throughout so the local steady-state temperature of the coolant T_c at radius r is related to the local disk temperature T by the expression

$$\frac{dT_c}{df} = \frac{2\pi hr}{M c_p}(T - T_c) \tag{3}$$

in which f denotes meridional length measured outwards along the curved surface of the disk. If the thermal conductivity k of the disk, which may vary with temperature, is treated for convenience as a function of r, then the steady-state heat conduction equation for the disk is

$$\frac{d^2T}{dr^2} + \left(\frac{1}{k}\frac{dk}{dr} + \frac{1}{r} + \frac{1}{b}\frac{db}{dr}\right)\frac{dT}{dr} - \frac{2h}{kb}\frac{df}{dr}(T - T_c) = 0. \tag{4}$$

We restrict the analytical solution to conditions in which M is large, and the thermal conductivity is independent of temperature, so that

$$T_c = T_{c0} \tag{5}$$

and

$$k = k_s \tag{6}$$

throughout, and, since the disk is comparatively thin,

$$\frac{df}{dr} = 1. \tag{7}$$

Equation (4) then becomes

$$\frac{d^2T}{dr^2} + \frac{n+1}{r}\frac{dT}{dr} - \frac{2h}{k_s b_s} r^{-n}(T - T_{c0}) = 0 \tag{8}$$

when b is written in terms of r from (1).

It is convenient to work in terms of dimensionless temperature ϕ and dimensionless radius s defined by

$$\phi = (T - T_{c0})/(T_1 - T_{c0}) \tag{9}$$

and

$$s = (r - r_0)/(r_1 - r_0) \tag{10}$$

so that (8) becomes

$$\frac{d^2\phi}{ds^2} + \frac{n+1}{A+s}\frac{d\phi}{ds} - C(A+s)^{-n}\phi = 0 \tag{11}$$

in which A and C are constants specified by the geometrical and thermal properties of the disk and by the heat-transfer coefficient, viz:

$$A = r_0/(r_1 - r_0) \tag{12}$$

$$C = 2h(r_1 - r_0)^{2-n}/kb_s. \tag{13}$$

The solution of (11), obtained by Lommel's transformation [3], is

$$\phi = (A+s)^{-n/2}[c I_\nu(z) + dK_\nu(z)] \tag{14}$$

where

$$\nu = n/(n-2) \tag{15}$$

and

$$z = \frac{2}{2-n} C^{1/2}(A+s)^{1-n/2}. \tag{16}$$

Noting that

$$\frac{d\phi}{ds} = C^{1/2}(A+s)^{-n}[c I_{\nu-1}(z) - dK_{\nu-1}(z)] \tag{17}$$

c and d are found from the conditions $\phi_1 = 1$ and $(d\phi/ds)_0 = 0$ to be

$$c = \frac{(A+1)^{n/2} K_{\nu-1}(z_0)}{I_\nu(z_1) K_{\nu-1}(z_0) + I_{\nu-1}(z_0) K_\nu(z_1)} \tag{18}$$

and

$$d = \frac{c I_{\nu-1}(z_0)}{K_{\nu-1}(z_0)}. \tag{19}$$

Typical results for the case of $n = 0$ (flat plate disk) and $n = -1$ (hyperbolic disk) are shown for a few values of A and C in Fig. 2, from which we see that the overall temperature distribution is comparatively insensitive to disk shape. The difference between the curves at the hub for the

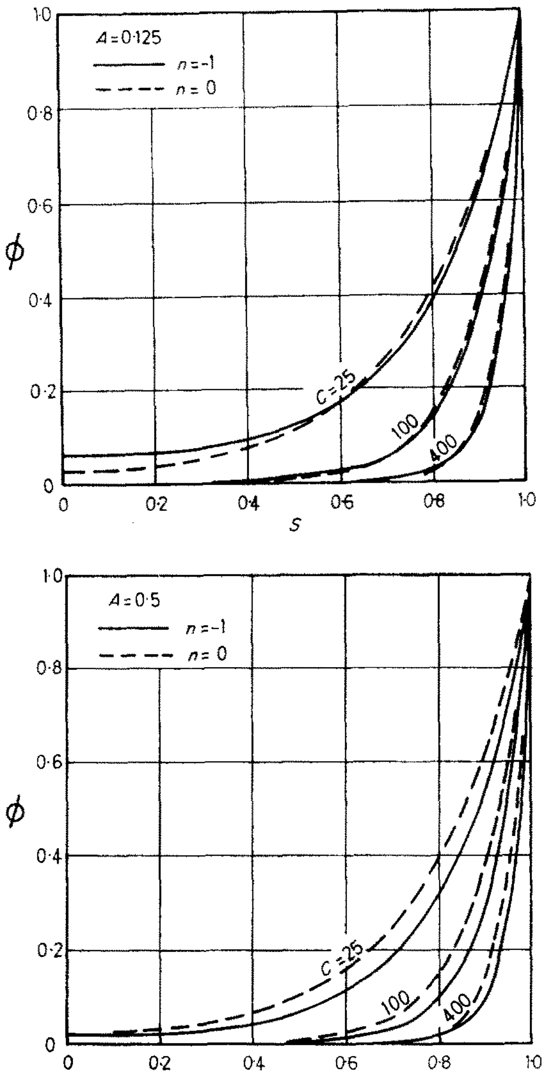

FIG. 2. Temperature distributions in disks of constant thermal conductivity with constant coolant temperature.

disk of tip/hub radius ratio of 9 ($A = 0{\cdot}125$) could be significant in applications where there is a limitation on hub temperature; the larger value of ϕ at the hub of a hyperbolic disk persists under conditions of reduced mass flow.

NUMERICAL SOLUTION OF THE HEAT FLOW EQUATIONS

In many practical cases the supply of coolant is restricted to such a degree that it experiences a considerable temperature rise as it passes over the surface of the disk. Moreover, the temperature range from the rim to the hub of the disk may be sufficiently wide for there to be appreciable changes in thermal conductivity of the disk material. A numerical solution of the coolant and disk temperature, equations (3) and (4), is adopted in these circumstances; since it is possible to allow for the difference between meridional length f measured along the curved surface of the disk and corresponding radial dimension r, this effect is also included. The heat-transfer coefficient h may now be regarded as an arbitrary function of radius. The restrictions of (5), (6) and (7) no longer apply.

Suppose that the thermal conductivity k of the disk material at temperature T is given by

$$k = k_s (1 + \mu T) \tag{20}$$

in which k_s is the conductivity at a reference temperature here taken as zero and μ is the temperature coefficient. (3) and (4) may conveniently be written in terms of dimensionless temperatures θ and ϕ and dimensionless radius s in the form

$$\frac{d\theta}{ds} = E(A + s)\{1 + [B(A+s)^{n-1}]^2\}^{1/2}(\phi - \theta) \tag{21}$$

and

$$\frac{d^2\phi}{ds^2} + \frac{D}{1 + D\phi}\left(\frac{d\phi}{ds}\right)^2 + \frac{n+1}{A+s}\frac{d\phi}{ds} - \frac{C(A+s)^{-n}}{1 + D\phi}\{1 + [B(A+s)^{n-1}]^2\}^{1/2}(\phi - \theta) = 0 \tag{22}$$

in which

$$\theta = \frac{T_c - T_{c0}}{T_1 - T_{c0}} \tag{23}$$

and ϕ and s are defined in (9) and (10). The constants appearing in (21) and (22) are specified by the properties of the disk and coolant, and by the heat-transfer coefficient, viz:

$$A = r_0/(r_1 - r_0) \tag{24}$$

$$B = \tfrac{1}{2} b_s n (r_1 - r_0)^{n-1} \tag{25}$$

$$C = 2h(r_1 - r_0)^{2-n}/k_s b_s (1 + \mu T_{c0}) \tag{26}$$

$$D = \mu (T_1 - T_{c0})/(1 + \mu T_{c0}) \tag{27}$$

$$E = 2\pi h (r_1 - r_0)^2 / M c_p. \tag{28}$$

(21) and (22) may be integrated numerically from $s = 0$ to $s = 1$ by a step-by-step method such as Runge–Kutta's. The known boundary conditions are

$$\dot{\phi} = \theta = 0 \quad \text{at} \quad s = 0 \tag{29}$$

and

$$\phi = 1 \quad \text{at} \quad s = 1 \tag{30}$$

so the solution is started with an estimated value of ϕ at $s = 0$. The integration is made to $s = 1$ where the computed value of ϕ is compared with (30). From this comparison a new starting value of ϕ is estimated and the procedure repeated as necessary.

Some results for typical values of the controlling parameters are presented in Figs. 3 and

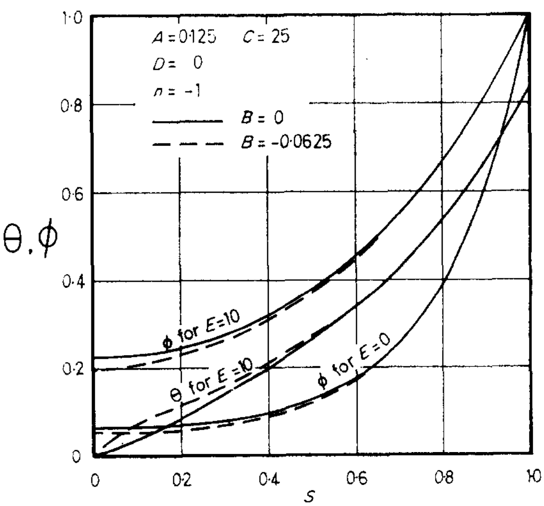

FIG. 3. Temperature distributions in disks of constant thermal conductivity showing effect of slope of disk surface.

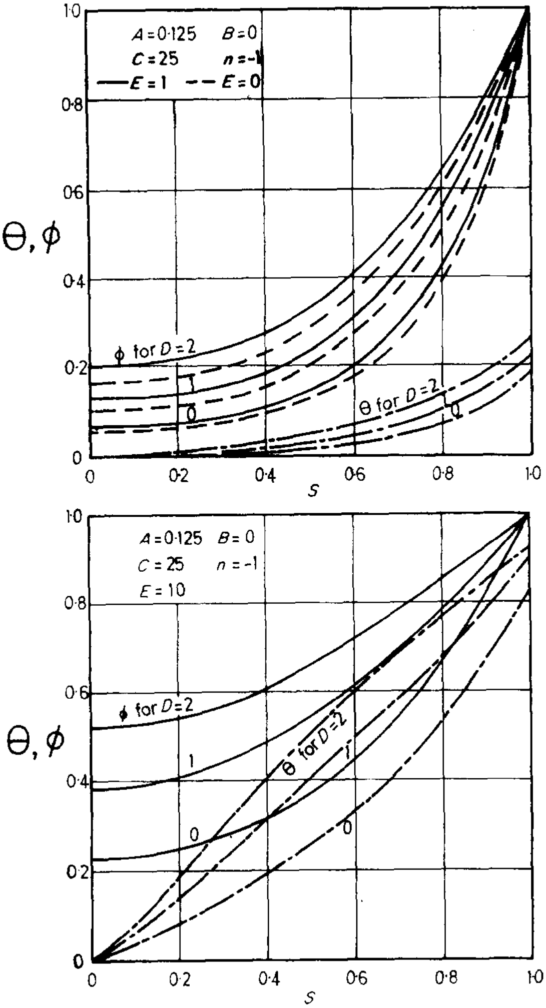

FIG. 4. Temperature distributions in disks showing effect of reducing coolant mass flow.

4. The effect of neglecting the difference between meridional length measured along the surface of the disk and the corresponding radial dimension is shown in Fig. 3. The values $n = -1$, $A = 0{\cdot}125$, $B = -0{\cdot}0625$ specify a disk in which the thickness at the hub is eight times the hub radius. For these proportions, temperature variation through the disk will not be negligible, so the comparatively small difference between the curves for $B = 0$ and $B = -0{\cdot}0625$ is inconsequential. Thus in any disk which is sufficiently thin to allow one-dimensional treatment of temperature the effect of slope of the surfaces may safely be neglected. Fig. 4 shows the effect of restricting coolant mass flow for a disk of typical proportions. For values of E below unity the temperature rise of the coolant is not more than about $\frac{1}{4}$ of the difference between disk rim temperature and coolant inlet temperature, and the distribution of temperature in the disk is almost the same as with infinite supply of coolant. Taking a disk for which $r_1 = 9$ in, $r_0 = 1$ in, $b_s = 1$ in and $k_s = 12$ chu/h ft degC, and a coolant for which $h = 30$ chu/h ft^2 degC and $c_p = 0{\cdot}25$ chu/lb degC (these being typical of a gas turbine application), the value $E = 1$ corresponds to a mass flow $M = 0{\cdot}09$ lb/s over each side of the disk, showing that a comparatively small flow provides quite effective cooling. The influence of variation of thermal conductivity with temperature is surprisingly large, although it should be noticed that the effect is due in part to the mean temperature in the disk being above the reference temperature for k_s. The values $D = 2$ and $D = 0$ correspond approximately to two typical disk steels (Nimonic 90 and F. V. 448 respectively) working over their useful temperature ranges.

UNSYMMETRICAL COOLING

In some cases it may not be possible to divide the coolant equally between the two surfaces of the disk, so that it is necessary to work in terms of mass flow rates, say M_r and M_l, over the right- and left-hand surfaces, which may now have different heat-transfer coefficients h_r and h_l. The heat flow equations become

$$\frac{dT_{cr}}{df} = \frac{2\pi h_r}{M_r c_p} r\,(T - T_{cr}) \tag{31}$$

$$\frac{dT_{cl}}{df} = \frac{2\pi h_l}{M_l c_p} r\,(T - T_{cl}) \tag{32}$$

$$\frac{d^2T}{dr^2} + \left(\frac{1}{k}\frac{dk}{dr} + \frac{1}{r} + \frac{1}{b}\frac{db}{dr}\right)\frac{dT}{dr} - \frac{h_r}{kb}\frac{df}{dr}(T - T_{cr}) - \frac{h_l}{kb}\frac{df}{dr}(T - T_{cl}) = 0. \tag{33}$$

The extension of the numerical integration procedure for a symmetrically cooled disk to this case is straightforward. It may be noted that the disk and coolant temperature distributions will not, in general, correspond with the symmetrical case, since for this to be so, comparison of (3) and (4) with (31), (32) and (33) requires

$$2h = h_r + h_l \tag{34}$$

$$\frac{h_r}{M_r} = \frac{h_l}{M_l}. \tag{35}$$

It is improbable that both these conditions would be satisfied simultaneously in practice.

ACKNOWLEDGEMENTS

The authors' thanks are due to Bristol-Siddeley Engines Ltd. for permission to publish the paper.

REFERENCES

1. A. E. JOHNSON, Turbine discs for jet propulsion units, *Aircr. Engng.* **28**, 187–195, 235–243, 265–272 (1956).
2. C. W. BERT, Non symmetric temperature distributions in varying-thickness circular fins, *J. Heat Transfer, Trans. A.S.M.E.* **85**, C, No. 1, 77–78 (1963).
3. G. N. WATSON, *Theory of Bessel Functions.* Cambridge University Press (1958).

Résumé—On a calculé la distribution de température à symétrie de révolution pour des disques dont l'épaisseur varie avec le rayon suivant une loi en puissance et en négligeant la variation de température dans l'épaisseur. On a trouvé une solution analytique dans le cas de disques de conductibilité thermique uniforme avec un coefficient de transfert de chaleur constant et une augmentation négligeable de la température du fluide refroidisseur. On a adopté une solution lorsque la conductivité varie avec le rayon ou encore lorsque l'augmentation de température du fluide refroidisseur est impossible. On a présenté quelques résultats de calcul montrant que des débits relativement faibles de fluide refroidisseur produisent un refroidissement effectif et que la variation de conductibilité thermique avec la température est sensible pour quelques matériaux d'usage courant.

Zusammenfassung—Die achssymmetrische Temperaturverteilung wird für Scheiben mit exponentiell vom Radius abhängiger Dicke berechnet, unter Vernachlässigung der Temperaturänderung durch die Dicke. Für Scheiben gleichmässiger Wärmeleitfähigkeit bei konstantem Wärmeübergangskoeffizienten und vernachlässigbarer Temperaturerhöhung des Kühlmittels liess sich eine analytische Lösung finden. Für temperaturabhängige Wärmeleitfähigkeit oder Änderung des Wärmeübergangskoeffizienten mit dem Radius oder einer merklichen Erhöhung der Külhmitteltemperatur wurde eine Lösung angenommen. Einige Rechenergebnisse werden angegeben; sie zeigen, dass schon relativ kleine Kühlmittelströme wirksam kühlen und dass temperaturabhängige Wärmeleitfähigkeit kennzeichnend für einige gegenwärtig verwendete Materialien ist.

Аннотация—Произведен расчет осесимметрического распределения температур для дисков, толщина которых изменяется экспоненциально с радиусом; изменениями температуры по толщине пренебрегают. Найдено аналитическое решение для случая дисков с постоянными вдоль них коэффициентами теплопроводности, постоянным коэффициентом теплообмена и пренебрежимо малым нагреванием охладителя. Указаны решения в случаях изменения теплопроводности с температурой, коэффициента теплообмена с радиусом или же значительного повышения температуры охладителя. Приводятся некоторые результаты расчетов, которые показывают, что относительно малые потоки охладителя дают значительное охлаждение и что изменение теплопроводности с изменением температуры значительно для некоторых материалов, используемых в настоящее время.